1. Motivation

The main goal of this talk was to give an introduction to the geometry of affine flag varieties and affine Grassmannians. I realized that some motivation for considering these objects is needed. So the first half of the talk will be the motivation for the second half. First half should be rather easy to understand and the second half will be more complicated and fast. In the first half we talk about representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and discuss (finite) Hecke algebras as convolution algebras. Second part discusses the same picture but in the affine setting. First part of the talk partly follows [Lo, Lecture 8].

We start from the following question. Pick a finite field $\mathbb{F}_q$, number $n \in \mathbb{Z}_{\geqslant 1}$ and consider the group $\mathrm{GL}_n(\mathbb{F}_q)$ of $n \times n$ invertible matrices with coefficients from $\mathbb{F}_q$. Group $\Gamma := \mathrm{GL}_n(\mathbb{F}_q)$ is finite. It then follows from the general theory that every finite dimensional representation of Γ over $\mathbb{C}$ is the direct sum of irreducible representations and irreducible representations are parametrized by conjugacy classes in Γ. The natural question is the following: explicitly construct an identification between conjugacy classes of Γ and irreducible representations of Γ i.e. starting from a conjugacy class of Γ to construct explicitly some irreducible representation of Γ.

Remark 1.1. *Recall that for example for symmetric groups S_n such a construction exists. Conjugacy classes are parametrized by partitions of n. The corresponding modules are called Specht modules. There is a classical approach to construct such modules and also there is a very nice approach due to Okounkov-Vershik see* [OV], [OV2].

In this talk we only consider conjugacy classes of unipotent elements of $\mathrm{GL}_n(\mathbb{F}_q)$ (recall that an element g is unipotent if $(g-1)^N = 0$ for large enough N). There is a bijection between conjugacy classes of unipotent elements and partitions of n. This follows from the following form the Jordan normal form theorem.

Proposition 1.2. *For any field $\mathbb{F}$ the operator $x \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ can be written in the Jordan normal form iff every eigenvalue of x (considered as an element of the algebraic closure $\overline{\mathbb{F}}$) lies in $\mathbb{F}$.*

Proof. Same proof as the one of the Jordan normal form theorem over $\mathbb{C}$. □

So we are considering only unipotent conjugacy classes. They should correspond to a certain subset of the set of irreducible representations of Γ. This subset consists of the representations V of Γ such that $V^B \neq \varnothing$, here $D \subset \mathrm{GL}_n(\mathbb{F}_q)$ is the subgroup of upper-triangular matrices (this is not an obvious statement, this section will be devoted to the explanation of this fact).

Remark 1.3. *Note that the condition that $V^D \neq \varnothing$ has to do with the representation theory of the Lie group $G = \mathrm{GL}_n(\mathbb{C})$ over complex numbers (vector of V^D is highest weight of weight zero). Note that for $G = \mathrm{GL}_n(\mathbb{C})$ the only irreducible finite dimensional representation V such that $V^D \neq \varnothing$ is the trivial representation.*

Let us now classify the representations V such that $V^D \neq \varnothing$. Let us first of all note that since D is a finite group its representations over $\mathbb{C}$ are completely reducible and this implies the equality $(V^*)^D \xrightarrow{\sim} (V^D)^*$. Recall also that

$$\mathbb{C}[\Gamma] \simeq \bigoplus_{V\text{-irreducible}} V \otimes V^* \tag{1.1}$$

and this is an isomorphism of $\Gamma\times\Gamma$-modules, here $\mathbb{C}[\Gamma]$ is the $\mathbb{C}$-vector space of functions $\Gamma \to \mathbb{C}$. The action of $\Gamma \times \Gamma$ on the LHS is given by $((g_1, g_2) \cdot f)(g) = f(g_1^{-1} g g_2)$ and the action of $\Gamma \times \Gamma$ on the RHS is just via $(g_1, g_2) \cdot (v \otimes v^\vee) = g_1 v \otimes g_2 v^\vee$.

Remark 1.4. *Recall that the isomorphism (1.1) can be constructed as follows: an element $v \otimes v^\vee \in V \otimes V^*$ maps to the function $g \mapsto \langle v, g v^\vee \rangle$.*

We can now consider the space of functions on Γ that are invariant with respect to D acting via right multiplications. This space is nothing else but $\mathbb{C}[\Gamma/D]$. From (1.1) we conclude that

$$\mathbb{C}[\Gamma/D] = \bigoplus_{V\text{-irreducible}} V \otimes (V^D)^*$$

so using Schur lemma we obtain

$$\mathrm{End}_\Gamma(\mathbb{C}[\Gamma/D]) = \bigoplus_{V\text{-irreducible }\Gamma\text{-module},\, V^D \neq \varnothing} \mathrm{End}_{\mathbb{C}}((V^D)^*).$$

We conclude that there is a bijection between irreducible representations of the (semisimple) algebra $\mathcal{H}_q := \mathrm{End}_\Gamma(\mathbb{C}[\Gamma/D])$ and the irreducible representations V of Γ such that $V^D \neq \varnothing$. Moreover this bijection is explicit. Starting from an irreducible representation E of $\mathcal{H}_q$ we construct the corresponding representation of Γ as follows. Note that we have the natural action $\mathcal{H}_q \curvearrowright \mathbb{C}[\Gamma/D]$ then

$$V = \mathrm{Hom}_{\mathcal{H}_q}(\mathbb{C}[\Gamma/D], E),$$

an action of Γ on V is induced from the action of Γ on Γ/D via left multiplication.

So our problem reduces to the classification of irreducible representations of the algebra $\mathcal{H}_q = \mathrm{End}_\Gamma(\mathbb{C}[\Gamma/D])$. To do this we need to understand the structure of the algebra $\mathcal{H}_q$. Let's try to describe this algebra via generators and relations. To do so we first study the set Γ/D in more details.

Set $G := \mathrm{GL}_n$, $W := S_n$ and let $B \subset G$ be the subgroup of upper-triangular matrices. We also denote by $U \subset B$ the subgroup of strictly upper-triangular matrices and by $T \subset B$ the subgroup of diagonal matrices. Note that $B = T \ltimes U$. Note also that $\Gamma = G(\mathbb{F}_q)$, $D = B(\mathbb{F}_q)$.

Remark 1.5. *More generally G is any reductive algebraic group, B is a Borel subgroup of G and W is the Weyl group of G.*

Set $\mathfrak{B} := G/B$. This is a (projective) algebraic variety of dimension $\frac{n(n-1)}{2}$. Note that Γ/H is nothing else but the set $\mathfrak{B}(\mathbb{F}_q)$ (see Remark 1.13). The variety $\mathfrak{B}$ is called the *flag variety* of G because of the following proposition.

Proposition 1.6. *The variety $\mathfrak{B}$ identifies with the space of flags:*

$$\mathfrak{B} \xrightarrow{\sim} \{0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = \mathbb{C}^n \mid \dim F_i = i\}.$$

The identification sends $[g] \in G/B$ to the flag $0 \subset g(\langle e_1 \rangle) \subset \ldots g(\langle e_1, \ldots, e_{n-1} \rangle) \subset \mathbb{C}^n$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{C}^n$.

Proof. Exercise. □

The following fact is standard and is known as Gauss decomposition.

Proposition 1.7. *We have*

$$G = \bigsqcup_{w \in W} BwB$$

Example 1.8. For $G = \mathrm{GL}_2$ we have $W = S_2 = \{1, (12)\}$ then we have

$$G = B \sqcup B(12)B,$$

where $B(12)B$ consists of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \neq 0$. Indeed for $c \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{a}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & b - \frac{ad}{c} \end{pmatrix}$$

As a corollary we obtain the following decomposition

$$\mathfrak{B} = \bigsqcup_{w \in W} BwB/B = \bigsqcup_{w \in W} UwB/B \tag{1.2}$$

that is known as Bruhat decomposition. We set $\mathfrak{B}_w := BwB/B$.

Remark 1.9. *Note that the subset $\{wB/B \mid w \in W\} \subset \mathfrak{B}$ is nothing else but the set $\mathfrak{B}^T$ of T-fixed points (interpretation of $\mathfrak{B}$ as the variety of flags helps to prove this) and each $\mathfrak{B}_w = UwB/B$ is nothing else but the attractor to wB/B with respect the $\mathbb{C}^\times$-action on $\mathfrak{B}$ via any cocharacter $\mathbb{C}^\times \to T$ given by $t \mapsto \mathrm{diag}(t^{k_1}, \ldots, t^{k_n})$ with $k_1 > \ldots > k_n$. So for the one who is familiar the Bruhat decomposition (1.2) can be considered as an example of Bialynicki-Birula decomposition (see* [Bia]*).*

Recall now that to every permutation $w \in S_n$ we can associate its length $l(w)$ that is equal to the number of disorders in w (i.e. pairs $1 \leqslant i < j \leqslant n$ such that $w(i) > w(j)$).

Proposition 1.10. *We have $\mathfrak{B}_w \simeq \mathbb{A}^{l(w)}$ i.e. each $\mathfrak{B}_w$ is a cell and the length function $l \colon W \to \mathbb{Z}_{\geqslant 0}$ has a geometric meaning as dimension of cells.*

Proof. Recall that $\mathfrak{B}_w = U \cdot wB/B$. Consider the subgroup U^w of U generated by matrices (a_{ij}) such that $a_{ij} = 0$ for $i < j$ such that $w(i) < w(j)$. We claim that the action of U^w on $\mathfrak{B}_w$ is free and transitive i.e. we have an isomorphism $U^w \xrightarrow{\sim} \mathfrak{B}_w$. To check that the action on w induces the isomorphism we can do the following. Pick a subgroup $U_w \subset U$ consisting of (a_{ij}) such that $a_{ij} = 0$ for $i < j$ such that $w(i) > w(j)$. We then claim that the multiplication map $U^w \times U_w \to U$ is the isomorphism. This follows easily from the Propoaition 4.1. Now we conclude that the action of U^w on $\mathfrak{B}_w$ is transitive. It follows that $\mathfrak{B}_w$ is a quotient of U^w by some subgroup of U^w. This subgroup is finite since the morphism $U^w \to \mathfrak{B}_w$ is an isomorphism at the level of tangent spaces $T_1 U^w \xrightarrow{\sim} T_{wB/B}\mathfrak{B}_w$. It follows that $\mathfrak{B}_w$ is a quotient of U^w by a finite group, hence is affine. It remains to note that the morphism $U^w \to \mathfrak{B}_w$ is $\mathbb{C}^\times$-equivariant with respect to the contracting $\mathbb{C}^\times$-actions. Proposition 4.1 finishes the proof. □

Let us now return to the algebra $\mathcal{H}_q = \mathrm{End}_\Gamma(\mathbb{C}[\Gamma/D])$. Recall that the group D is finite so the functors of taking invariants $\bullet^D$ and coinvariants $\bullet \otimes_{\mathbb{C}H} \mathbb{C}$ are isomorphic (here $\mathbb{C}H$ is the group algebra of H). We conclude that $\mathbb{C}[\Gamma/D] = \mathbb{C}[\Gamma]^D \simeq \mathbb{C}[\Gamma] \otimes_{\mathbb{C}H} \mathbb{C}$. By the Frobenius reciprocity we have

$$\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma/D], -) \simeq \mathrm{Hom}_D(\mathbb{C}, -).$$

It follows that $\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma/D], \mathbb{C}[\Gamma/D]) = \mathrm{Hom}_D(\mathbb{C}, \mathbb{C}[\Gamma/D]) = \mathbb{C}[\Gamma/D]^D$.

We are now ready to compute the dimension of $\mathcal{H}_q$.

Gauss decomposition is still valid for $\Gamma = G(\mathbb{F}_q)$, $D = B(\mathbb{F}_q)$ and gives us

$$\Gamma = \bigsqcup_{w \in W} DwD,\ \Gamma/D = \bigsqcup_{w \in W} DwD/D$$

so we conclude that $\dim \mathcal{H}_q = |W| = |S_n| = n!$. To describe the algebra structure on $\mathcal{H}_q$ we will identify it with another algebra.

Definition 1.11. Let $\mathcal{H}'_q := \mathbb{C}[D\backslash\Gamma/D] = \mathbb{C}[\mathfrak{B}(\mathbb{F}_q)]^D$. We define the algebra structure on $\mathcal{H}'_q$ using the so-called convolution product as follows:

$$(f_1 * f_2)(g) = \frac{1}{|B(\mathbb{F}_q)|} \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

Question to the audience: who is the identity in $\mathcal{H}'_q$?

Remark 1.12. *Let δ_1 be the function that is equal to* 1 *on $\mathfrak{B}_1(\mathbb{F}_q)$ and is zero otherwise i.e. δ_1 is the characteristic function of $\mathfrak{B}_1(\mathbb{F}_q)$, $\delta_1 = \chi_{\mathfrak{B}_1(\mathbb{F}_q)}$. Then for every D-invariant function f on $\mathfrak{B}(\mathbb{F}_q)$ we have*

$$f * \delta_1(g) = \frac{1}{|D|} \sum_{b \in D} f(gb^{-1}) \delta_1(b) = f(g)$$

and similarly

$$\delta_1 * f(g) = \frac{1}{|D|} \sum_{b \in D} \delta_1(b) f(b^{-1}g) = f(g).$$

So $\delta_1 \in \mathcal{H}'_q$ is the identity element.

So we obtain some algebra $\mathcal{H}'_q$. Note that $\dim \mathcal{H}'_q = n! = \dim \mathcal{H}_q$ and moreover we have a natural basis of $\mathcal{H}'_q$ consisting of characteristic functions $T_w = \chi_{\mathfrak{B}_w(\mathbb{F}_q)}$ of D-orbits.

Remark 1.13. *Note that identifying the realization of $\mathcal{H}'_q$ as $\mathbb{C}[B(\mathbb{F}_q)\backslash \mathrm{GL}_n(\mathbb{F}_q)/B(\mathbb{F}_q)]$ with its realization as $\mathbb{C}[\mathfrak{B}(\mathbb{F}_q)]^{B(\mathbb{F}_q)}$ we implicitly use the equality $\mathfrak{B}(\mathbb{F}_q) = G(\mathbb{F}_q)/B(\mathbb{F}_q)$. This can be shown using Gauss and Bruhat decompositions of $G(\mathbb{F}_q)$, $\mathfrak{B}(\mathbb{F}_q)$, more conceptual way to prove this is the following. Note that we have an exact sequence*

$$0 \to B(\mathbb{F}_q) \to G(\mathbb{F}_q) \to \mathfrak{B}(\mathbb{F}_q) \to H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), B(\overline{\mathbb{F}}_q)),$$

where $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), B(\overline{\mathbb{F}}_q))$ is the first Galois cohomology of the group $B(\overline{\mathbb{F}}_q)$. So our goal is to show that $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), B(\overline{\mathbb{F}}_q)) = 0$. Note now that B can be filtered by normal subgroups with successive quotients being either $\mathbb{G}_a$ or $\mathbb{G}_m$ and then the *claim follows from the Hilbert* 90 *theorem (which implies that* $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \overline{\mathbb{F}}_q^\times) = 0$*) together the fact that* $H^1(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q), \overline{\mathbb{F}}_q) = 0$ *(that is true since* $\mathbb{F}_q$ *is perfect).*

Proposition 1.14. *There is an isomorphism of algebras*

$$\mathcal{H}'_q \xrightarrow{\sim} \mathcal{H}_q \text{ given by } f \mapsto f * -.$$

Proof. We give an idea. The map is well-defined since $f * -\colon \mathbb{C}[\Gamma/D] \to \mathbb{C}[\Gamma/D]$ is Γ-equivariant by the definitions (easy computation, left as an exercise). Algebras have equal dimensions. The map is injective since the image of f sends $\delta_1 \in \mathbb{C}[\mathfrak{B}(\mathbb{F}_q)]$ to $f * \delta_1 = f$. Claim follows. □

Let us now describe the multiplication on $\mathcal{H}_q$ in the basis of characteristic functions of $B(\mathbb{F}_q)$-orbits that we denote by $\chi_w = T_w$.

Proposition 1.15. *We have*

$$T_{w_1}T_{w_2} = T_{w_1w_2} \text{ if } l(w_1w_2) = l(w_1) + l(w_2),$$

$$T^2_{(i,i+1)} = q + (q-1)T_{(i,i+1)}, i = 1, \dots, n-1.$$

Proof. The idea is the following. Pick $w, v \in W$ and let us write

$$T_wT_v = \sum_u m^u_{w,v} T_u$$

with $m^u_{w,v} \in \mathbb{C}$. Our goal is to compute the numbers $m^u_{w,v}$. Note that $m^u_{w,v}$ is nothing else but $(\chi_w * \chi_v)(u) = \frac{1}{|D|}\sum_{g_1 \in DwD,\, g_2 \in DvD,\, g_1g_2 = u} 1$. This can be rewritten as follows: $g_1 \in DwD$ and $g_1 = ug_2^{-1} \in uDv^{-1}D$ so the sum above is equal to $\frac{|DwD \cap uDv^{-1}D|}{|D|}$ i.e. is equal to the number of elements in $\mathfrak{B}_w(\mathbb{F}_q) \cap u\mathfrak{B}_{v^{-1}}(\mathbb{F}_q)$ i.e.

$$T_wT_v = \sum_u |\mathfrak{B}_w(\mathbb{F}_q) \cap u\mathfrak{B}_{v^{-1}}(\mathbb{F}_q)| T_u.$$

We can also assume that $v = s = (i, i+1)$. Note that $\mathfrak{B}_s = \mathbb{P}^1$. Computation of $|\mathfrak{B}_w(\mathbb{F}_q) \cap u\mathfrak{B}_s(\mathbb{F}_q)|$ is an exercise on the geometry of flag varieties. □

Example 1.16. Consider the example $G = \mathrm{GL}_2$. Then the only interesting computation is $T^2_{(12)}$. We have

$$T^2_{(12)} = |\mathfrak{B}_{(12)}(\mathbb{F}_q)| + |\mathfrak{B}_{(12)}(\mathbb{F}_q) \cap (12)\mathfrak{B}_{(12)}(\mathbb{F}_q)| T_{(12)}.$$

Recall now that by Proposition 1.10 we have $\mathfrak{B}_{(12)} = \mathbb{A}^1$ so $|\mathfrak{B}_{(12)}(\mathbb{F}_q)| = q$. It remains to compute the number of points in $\mathfrak{B}_{(12)}(\mathbb{F}_q) \cap (12)\mathfrak{B}_{(12)}(\mathbb{F}_q)$. The variety $\mathfrak{B}$ is isomorphic to $\mathbb{P}^1$, $\mathfrak{B}_{(12)} = \mathbb{P}^1 \setminus \mathfrak{B}_1$ is $\mathbb{P}^1 \setminus \{0\}$ and $(12)\mathfrak{B}_{(12)} = \mathbb{P}^1 \setminus (12)\mathfrak{B}_1$ is $\mathbb{P}^1 \setminus \{\infty\}$. So we conclude that the intersection $\mathfrak{B}_{(12)} \cap (12)\mathfrak{B}_{(12)}$ is $\mathbb{P}^1 \setminus \{0, \infty\}$ that has $q-1$ $\mathbb{F}_q$-points. We conclude that

$$T^2_{(12)} = q + (q-1)T_{(12)}$$

as desired.

The isomorphism $\mathbb{C}S_2 \xrightarrow{\sim} \mathcal{H}_q$ is given by

$$1 \mapsto 1,\ (12) \mapsto \frac{T_{(12)}}{\sqrt{1+q^2}} + \frac{1-q}{2\sqrt{1+q^2}},$$

it is well-defined iff $q^2 \neq -1$.

Remark 1.17. *Note that the relation $T_s^2 = q + (q-1)T_s$ can be rewritten as $(T_s+1)(T_s - q) = 0$.*

It is now easy to see that the relations from Proposition 1.15 determine the multiplication in algebra $\mathcal{H}'_q = \mathcal{H}_q$ uniquely and that this algebra is generated by the elements $T_{(i,i+1)}$, $i = 1, \ldots, n-1$. Now we can forget that q was a power of a prime and note that we obtain a new interesting algebra $\mathcal{H}_q$ for every $q \in \mathbb{C}$. This algebra is called a Hecke algebra corresponding to W. Note that for $q = 1$ we have $\mathcal{H}_1 = \mathbb{C}S_n$. Moreover if q is not a root of unity then the algebra $\mathcal{H}_q$ is *noncanonically* isomorphic to the group algebra $\mathbb{C}S_n$. In general $\mathcal{H}_q$ can be considered as a *flat deformation* of the algebra $\mathbb{C}S_n$. In the same spirit as the classical story about S_n one can construct explicitly Specht modules for the algebra $\mathcal{H}_q$. Note that the number of irreducible representations of $\mathcal{H}_q$ is the same as the number of partitions of n that is in bijection with unipotent conjugacy classes in $\Gamma = \mathrm{GL}_n(\mathbb{F}_q)$. This finishes our task of explicit construction of irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ corresponding to unipotent conjugacy classes. They are given by

$$V_A := \mathrm{Hom}_{\mathcal{H}_q}(\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)/B(\mathbb{F}_q)], S_A),$$

here S_A is the Specht module corresponding to the partition A of n.

The Hecke algebra $\mathcal{H}_q$ that we have just constructed is a very important algebra in representation theory. Recall now that $W = S_n$ is the Weyl group of $G = \mathrm{SL}_n$ or the Weyl group of $\mathfrak{sl}_n$. We just realized that the Hecke algebra of W can be constructed using the geometry of the flag variety $\mathfrak{B}$.

Remark 1.18. *Note that the flag variety $\mathfrak{B}$ of GL_n coincides with the one of SL_n and also with the one for PGL_n.*

The natural question is the following: what is the affine analog of this picture?

By affine analog I mean the following. Note that S_n is the Weyl group of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$, we can now consider the affine Lie algebra $\hat{\mathfrak{sl}}_n$ defined as follows:

$$\hat{\mathfrak{sl}}_n := \mathfrak{sl}_n((z)) \oplus \mathbb{C}K$$

with the commutator given by

$$[x \otimes z^k, y \otimes z^l] = [x,y] \otimes z^{k+l} + k\delta_{k+l,0}\,\mathrm{tr}(xy)K,\ [K, \hat{\mathfrak{g}}] = 0.$$

One can associate to $\hat{\mathfrak{sl}}_n$ its Weyl group $\hat{W}$. The group $\hat{W}$ can be described as follows. Set $\Lambda := \mathbb{Z}^n/\{(k,\ldots,k),\ k \in \mathbb{Z}\}$. Let $Q \subset \Lambda$ be the subgroup of Λ generated by the elements $(k_1, \ldots, k_n)$ with $k_1 + \ldots + k_n = 0$.

Remark 1.19. *It is an exercise to show that $[\Lambda : Q] = n$. The representatives of Λ/Q are classes of the elements $(1,\dots,1,0,\dots,0)$. Actually Q is the root lattice of $\mathfrak{sl}_n$ and Λ is the weight lattice of $\mathfrak{sl}_n$.*

It is not hard to show that $\hat{W} = S_n \ltimes Q$ and $\hat{W}$ is generated by the reflections

$$s_i = ((i,i+1),0), i = 1,\dots,n-1,\ s_0 = ((1,n),(1,0,\dots,0,-1)).$$

Remark 1.20. *Recall that the multiplication in $S_n \ltimes Q$ is given by $(w_1,\lambda_1)\cdot(w_2,\lambda_2) = (w_1w_2, w_2^{-1}(\lambda_1)+\lambda_2)$.*

Remark 1.21. *Group $\hat{W}$ is an example of the affine Coxeter group. It is generated by reflections $s_0,\dots,s_{n-1}$, defining relations are the following. It is more elegant to write them down using the identification $\{0,1,\dots,n-1\} \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}$, $i \mapsto \bar{i}$.*

(1) $s_{\bar{i}}^2 = 1$.
(2) $s_{\bar{i}}s_{\bar{j}} = s_{\bar{j}}s_{\bar{i}}$ *for* $\bar{j} \notin \{\overline{i-1},\bar{i},\overline{i+1}\}$.
(3) $s_{\bar{i}}s_{\overline{i+1}}s_{\bar{i}} = s_{\overline{i+1}}s_{\bar{i}}s_{\overline{i+1}}$ *for* $\bar{i} \in \mathbb{Z}/n\mathbb{Z}$.

Note that $s_{\bar{0}}$ and $s_{\bar{i}}$, $i = 1,\dots,n-1$ have different nature but it turns out that the defining relations are cyclically symmetric. Note that the third relation is equivalent to $(s_{\bar{i}}s_{\overline{i+1}})^3 = 1$.

Note also that since $\hat{W}$ is a Coxeter group then we have a length function $l\colon \hat{W} \to \mathbb{Z}_{\geqslant 0}$ that can be defined as follows: $l(\hat{w})$ is the number of reflections $s_{\bar{i}}$ that appear in a reduced decomposition of $\hat{w}$. This function can be also (partly) described as follows: $l(w)$ is the number of disorders of $w \in W$, $l(\lambda) = (2\rho,\lambda^+)$ where $\lambda^+ \in Q$ is such that $\lambda_1^+ \geqslant \dots \geqslant \lambda_n^+$ (i.e. λ^+ is dominant), here $2\rho = (n,n-1,\dots,1)$ and $(\ ,\ )$ corresponds to the standard pairing on Q. Also $l(w,\lambda) = l(w) + (2\rho,\lambda)$ for dominant $\lambda \in Q$.

We now want to define the algebra $\mathcal{H}_q(\hat{W})$ that should be the affine Hecke algebra (sometimes called the Iwahori-Hecke algebra or Iwahori-Matsumoto Hecke algebra) and try to find some geometric object that place a role of $\mathfrak{B}$ for $\mathcal{H}_q(\hat{W})$.

2. Affine Grassmannian and affine flag variety

Here $\mathfrak{g} = \mathfrak{sl}_n$ and G is the Lie group with Lie algebra $\mathfrak{g}$ (so either SL_n or PGL_n), sometimes we also denote GL_n by G.

Remark 2.1. *Everything that will be discussed in this talk can be generalized to arbitrary reductive Lie algebra $\mathfrak{g}$ and reductive algebraic group G with Lie algebra equal to $\mathfrak{g}$.*

Our goal is to define a "flag variety" corresponding to $\hat{\mathfrak{g}}$. Since we passed from $\mathfrak{g}$ to $\hat{\mathfrak{g}} = \mathfrak{g}((z)) \oplus \mathbb{C}K$ then the first idea is to pass from G to $G(\mathcal{K})$, where $\mathcal{K} = \mathbb{C}((z))$. Recall now that $\mathfrak{B}$ was the quotient of G by B and now we replace G by $G(\mathcal{K})$ and want to replace B by some subgroup of $G(\mathcal{K})$. I claim that we have a very natural candidate for such a subgroup.

Question to the audience: suggestions for the correct analog of B in this setting?

Note that $B \subset G$ is "the half" of G up to diagonal matrices. Let $\mathcal{O} \subset \mathbb{C}((z))$ be the ring $\mathbb{C}[[z]]$. Consider the subgroup $G(\mathcal{O}) \subset G(\mathcal{K})$. Note that $G(\mathcal{O})$ is almost the half of $G(\mathcal{K})$ up to the fact that for "$z = 0$" the group "$G(\mathbb{C}[[z]])_{z=0}$" is the whole G. We want for $z = 0$ to get B so we do the following: let $\mathrm{ev}_0 \colon G(\mathcal{O}) \to G$ be the evaluation at zero homomorphism. Set

$$I := \mathrm{ev}_0^{-1}(B) \subset G(\mathcal{K}).$$

The group I is called the Iwahori subgroup of $G(\mathcal{K})$.

Remark 2.2. *Recall that $B = T \ltimes U$. Now $U \subset G$ is the subgroup of G generated by positive root subgroups. For affine Lie algebras we can also easily define the notion of positive and negative roots and then the definition of I becomes transparent.*

We can now define

$$\mathrm{Fl}_G := G(\mathcal{K})/I,\ \mathrm{Gr}_G := G(\mathcal{K})/G(\mathcal{O})$$

The space Fl_G is called the *affine flag variety of* G and Gr_G is the *affine Grassmannian of* G.

Remark 2.3. *The space Fl_G should be considered as an affine analog of $\mathfrak{B}$, the space Gr_G does not have an interesting classical analog since its classical analog should be just $G/G = \mathrm{pt}$.*

Our goal for now is to study the space Fl_G and to use its geometry to obtain the affine Hecke algebra $\mathcal{H}_q(\hat{W})$. We will also study the geometry of Gr_G and describe the corresponding (spherical) Hecke algebra.

Let us first of all note that we have a surjective map $\mathrm{Fl}_G \to \mathrm{Gr}_G$ which fibers are $G(\mathcal{O})/I = G/B = \mathfrak{B}$. So Fl_G can be considered as a fiber bundle over Gr_G with fibers isomorphic to $\mathfrak{B}$. So we see that the geometry of Fl_G is the "combination" of the geometries of Gr_G and $\mathfrak{B}$. We concentrate on the case $G = \mathrm{GL}_n$ since the affine Grassmanian and flags of SL_n, PGL_n can be easily extracted from the same varieties for GL_n (see Remark 2.5).

Our first goal is to understand the geometric structure of $\mathrm{Fl}_G, \mathrm{Gr}_G$. We concentrate on Gr_G since Fl_G is just the fiber bundle over Gr_G. Let V be a vector space of dimension n. The following proposition identifies $\mathrm{Gr}_{\mathrm{GL}_n}$ with the space of $\mathcal{O}$-lattices in $V(\mathcal{K}) := V \otimes_{\mathbb{C}} \mathcal{K}$ (free rank n submodules).

Proposition 2.4. *For $G = \mathrm{GL}_n$ we have*

$$\mathrm{Gr}_{\mathrm{GL}_n} = \{L \subset V(\mathcal{K}) \mid L \text{ is a finitely generated } \mathcal{O}\text{-submodule that generates } V(\mathcal{K}) \text{ over } \mathcal{K}\}$$

The description of $\mathrm{Fl}_{\mathrm{GL}_n}$ is the following. We have

$$\mathrm{Fl}_{\mathrm{GL}_n} = \{(L, F_\bullet) \mid L \text{ is } \mathcal{O}\text{-lattice in } V(\mathcal{K}),\ F_\bullet \text{ is a flag in } L/zL\}.$$

Equivalently we have

$$\mathrm{Fl}_{\mathrm{GL}_n} = \{L = L_1 \supset L_2 \supset \ldots \supset L_n \supset zL \mid L_i\text{-lattices}\}$$

that can be considered as an infinite flag if we set $z^k L_i := L_{kn+i}$.

Proof. We prove the claim for Gr_G. Recall that $\mathrm{Gr}_{\mathrm{GL}_n} = \mathrm{GL_n}(\mathcal{K})/\mathrm{GL}_n(\mathcal{O})$. Note that we have a lattice $L_0 := V(\mathcal{O}) \subset V(\mathcal{K})$. I claim that every lattice $L \subset V(\mathcal{K})$ can be obtained as $g(L_0)$ for an appropriate $g \in \mathrm{GL}_n(\mathcal{K})$. Indeed recall that $L \subset V(K)$ is a finitely generated $\mathcal{O} = \mathbb{C}[[z]]$-submodule of $V(\mathcal{K})$ that must be free since $\mathbb{C}[[t]]$ is PID and $V(\mathcal{K})$ has no torsion. Since $\dim_{\mathcal{K}} L \otimes_{\mathcal{O}} \mathcal{K} = n$ we conclude that L is a free rank n $\mathcal{O}$-module. We can then fix some generators $v_1, \ldots, v_n$ of L and consider $g \in \mathrm{GL}_n(\mathcal{K})$ that sends e_i to v_i.

Note now that an element $g \in \mathrm{GL}_n(\mathcal{K})$ stabilizes L_0 (i.e. $g(L_0) = L_0$) iff $g \in \mathrm{GL}_n(\mathcal{O})$. This finishes the proof of the first claim of the proposition. $\square$

Remark 2.5. *Let us describe* $\mathrm{Gr}_{\mathrm{SL}_n}$, $\mathrm{Gr}_{\mathrm{PGL}_n}$ *via lattices. Recall that we have the "standard" lattice* L_0. *For an arbitrary lattice we define the index* $[L : L_0]$ *as* $|L/(L \cap L_0)| - |L_0/(L \cap L_0)|$ *(note that for* $L \supset L_0$ *we have* $[L : L_0] = |L/L_0|$*). Then* $\mathrm{Gr}_{\mathrm{SL}_n}$ *consists of lattices* L *such that* $[L : L_0] = 0$. *The natural embedding* $\mathrm{Gr}_{\mathrm{SL}_n} \subset \mathrm{Gr}_{\mathrm{GL}_n}$ *realizes* $\mathrm{Gr}_{\mathrm{SL}_n}$ *as the connected component of* $L_0 \in \mathrm{Gr}_{\mathrm{GL}_n}$. *Note that* $\mathrm{Gr}_{\mathrm{GL}_n}$ *has* $\mathbb{Z}$ *connected components, the connected component that corresponds to* $k \in \mathbb{Z}$ *consists of lattices* L *such that* $[L : L_0] = k$, *the basic example of such lattice is the one generated by* $\{z^k e_1, e_2, \ldots, e_n\} \in V(\mathcal{K})$. *Actually more generally* $\pi_0(\mathrm{Gr}_G) = \pi_1(G)$.

Let us now describe the affine Grassmannian $\mathrm{Gr}_{\mathrm{PGL}_n}$. *We have the natural surjection* $\mathrm{Gr}_{\mathrm{GL}_n} \twoheadrightarrow \mathrm{Gr}_{\mathrm{PGL}_n}$. *It identifies* $\mathrm{Gr}_{\mathrm{PGL}_n}$ *with the space of lattices* L *modulo the relation* $L \sim zL$. *Note now that to every such equivalence class* $\overline{L}$ *one can associate the index* $[L : L_0]$ *that is a well-defined element of* $\mathbb{Z}/n\mathbb{Z}$. *We see that* $\mathrm{Gr}_{\mathrm{PGL}_n}$ *has* n *connected components (compare with the fact that* $\pi_1(\mathrm{PGL}_n) \simeq \mathbb{Z}/n\mathbb{Z}$*).*

We are now ready to understand the structure of $\mathrm{Gr}_{\mathrm{GL}_n}$. For $N \in \mathbb{Z}_{\geqslant 1}$ set

$$\mathrm{Gr}^N_{\mathrm{GL}_n} = \{L \in \mathrm{Gr}_G \mid z^N L_0 \subset L \subset z^{-N} L_0\}.$$

It is clear that $\mathrm{Gr}_G = \bigcup_N \mathrm{Gr}^N_G$. I claim that Gr^N_G is a closed subvariety of the disjoint union $\mathrm{Gr}(\bullet, 2nN) := \bigsqcup_{k \in \mathbb{Z}_{\geqslant 0}} \mathrm{Gr}(k, 2nN)$. Indeed we have an embedding $\mathrm{Gr}^N_G \hookrightarrow \mathrm{Gr}(\bullet, 2nN)$ that sends L to $L/z^N L_0$. Moreover the image of this embedding consists exactly of subspaces $R \subset z^{-N} L_0/z^N L_0$ that are invariant with respect to the operator $z\cdot \colon z^{-N} L_0/z^N L_0 \to z^{-N} L_0/z^N L_0$ that is indeed a closed condition.

So we see that Gr_G should be considered as an inductive limit of projective schemes Gr^N_G of finite type. Same holds for Fl_G. In other words Gr_G is an ind-scheme.

Being more careful we should consider Gr_G as a functor $\mathrm{Alg}_{\mathbb{C}} \to \mathrm{Sets}$ defined as follows. By the definition we say that $L \subset R((z))^{\oplus n}$ is a lattice if L is a finitely generated projective $R[[z]]$-module such that $L \otimes_{R[[z]]} R((z)) = R((z))^{\oplus n}$. Then

$$\mathrm{Gr}_{\mathrm{GL}_n} \colon \mathrm{Alg}_{\mathbb{C}} \to \mathrm{Sets},\ R \mapsto \{L \subset R((z))^{\oplus n} \mid L \text{ is a lattice}\}.$$

Then the functor $\mathrm{Gr}_{\mathrm{GL}_n}$ is represented by an ind-scheme.

Remark 2.6. *One can show that the functors* Gr_G, Fl_G *are not represented by schemes when* G *is nontrivial. Let us explain this for* $G = \mathbb{G}_a$ *i.e.* $G = \mathbb{A}^1$ *with the group structure given by addition.*

Using the realization of $\mathrm{Gr}_{\mathbb{G}_a}$ *as* $\mathbb{G}_a(\mathcal{K})/\mathbb{G}_a(\mathcal{O})$ *we see that* $\mathrm{Gr}_{\mathbb{G}_a}(\mathbb{C})$ *is* $\mathbb{C}((z))/\mathbb{C}[[z]] = z^{-1}\mathbb{C}[z^{-1}] = \bigcup_{k \in \mathbb{Z}_{\geqslant 0}} \mathrm{Span}_{\mathbb{C}}(z^{-1}, \ldots, z^{-k})$ *and more generally* $\mathrm{Gr}_{\mathbb{G}_a}(R) = R((z))/R[[z]] = z^{-1}R[z^{-1}]$ *i.e.* $\mathrm{Gr}_{\mathbb{G}_a}$ *is the inductive limit* $\varinjlim \mathbb{A}^k$. *So for a* $\mathbb{C}$-*algebra* R *we have* $\mathrm{Gr}_{\mathbb{G}_a}(R) = \varinjlim \mathbb{A}^k(R)$.

If $\mathrm{Gr}_{\mathbb{G}_a}$ *is represented by a scheme then we can find* $U \subset \mathrm{Gr}_{\mathbb{G}_a}$ *an affine open subscheme of* $\mathrm{Gr}_{\mathbb{G}_a}$. *We have* $U = \operatorname{Spec} R$ *and the map* $U \to \mathrm{Gr}_{\mathbb{G}_a}$ *corresponds to* *some element of* $\mathrm{Gr}_{\mathbb{G}_a}(R)$ *i.e. an element of some* $\mathbb{A}^k(R)$ *i.e. the image of* U *lies in* $\mathbb{A}^k \subset \mathrm{Gr}_{\mathbb{G}_a}$. *It remains to note that the embedding* $\mathbb{A}^k \subset \mathrm{Gr}_{\mathbb{G}_a}$ *is closed so the composition* $U \subset \mathbb{A}^k \subset \mathrm{Gr}_{\mathbb{G}_a}$ *can not be open.*

We are now ready to formulate the analog of Bruhat decompositions for Fl_G, Gr_G. Assume that $G = \mathrm{SL}_n$. Recall the subgroup $T \subset \mathrm{SL}_n$ of diagonal matrices. Recall also the lattice Q consisting of $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that $k_1 + \ldots + k_n = 0$. Let $Q^+ \subset Q$ be the submonoid of Q consisting of elements $(k_1, \ldots, k_n)$ such that $k_1 \geqslant \ldots \geqslant k_n$. To any $\lambda = (k_1, \ldots, k_n)$ we can associate the following point of $T(\mathcal{K})$: $z^\lambda := \mathrm{diag}(z^{k_1}, \ldots, z^{k_n})$. We denote by the same symbol z^λ the corresponding points of Gr_G, Fl_G.

The following proposition is a version of a Bruhat decomposition in the affine setting.

Proposition 2.7. *We have*

$$\mathrm{Fl}_{\mathrm{SL}_n} = \bigsqcup_{(w,\mu) \in \hat{W}} Iwz^\mu, \; \mathrm{Gr}_{\mathrm{SL}_n} = \bigsqcup_{\lambda \in Q^+} \mathrm{SL}_n(\mathcal{O}) z^\lambda.$$

Variety $\mathrm{Fl}_G^{(w,\mu)} := Iwz^\mu$ *is isomorphic to* $\mathbb{A}^{l(wz^\mu)}$ *(in particular finite dimensional), where* $l \colon \hat{W} \to \mathbb{Z}_{\geqslant 0}$ *is the length function. Variety* $\mathrm{Gr}_G^\lambda := G(\mathcal{O}) z^\lambda$ *is smooth of dimension* $l(\lambda) = (2\rho, \lambda)$, *here* $2\rho = (n, n-1, \ldots, 2, 1)$.

Proof. We (partly) prove the claim for Gr_G. The claim for Fl_G can be deduced using the $\mathfrak{B}$-fibration $\mathrm{Fl}_G \to \mathrm{Gr}_G$. We also assume that $G = \mathrm{GL}_n$ for simplicity (for $G = \mathrm{GL}_n$ group $\hat{W}$ should be replaced by $S_n \ltimes \mathbb{Z}^n$ and Q^+ by $\{(k_1, \ldots, k_n) \in \mathbb{Z}^n\}$ such that $k_1 \geqslant \ldots \geqslant k_n$).

Recall the lattice description of Gr_G. In terms of lattices points z^λ correspond to lattices generated by vectors $z^{k_1} e_1, \ldots, z^{k_n} e_n \in V(\mathcal{K})$. Consider now any lattice $L \subset V(\mathcal{K})$. Let $v_1, \ldots, v_n$ be generators of L. We can assume that among $\{v_i\}$ v_1 has the greatest pole at zero say equal to $k \in \mathbb{Z}$. Reordering $\{e_i\}$ we can write $v_1 = z^k v'$ with $v' = \sum_i a_i e_i$, $a_i \in \mathbb{C}[[z]]$, $a_1 \in \mathbb{C}[[z]]^\times$. Using the element of $\mathrm{GL}_n(\mathcal{O})$ that sends e_1 to $a_1^{-1} e_1$ and e_i to e_i for $i > 1$ we can then assume that $v_1 = z^k(e_1 + \sum_{i>1} a_i e_i)$. Then we can apply the authomorphism that sends $v' = e_1 + \sum_{i>1} a_i e_i$ to e_1 and e_i to e_i for $i > 1$ and conclude that v_1 becomes $z^k e_1$. After that we can change our basis elements $v_2, \ldots, v_n$ $\mathrm{GL}_n(\mathcal{O})$ of L and delete e_1 from the decompositions of v_i, $i > 1$ (use that $z^k e_1$ lies in the new lattice). We then conclude that $v_1 = z^k e_1$ and $v_i \in \mathrm{Span}_{\mathbb{C}}(e_2, \ldots, e_n) \otimes_{\mathbb{C}} \mathcal{K}$. We repeat the procedure for the lattice in $\mathrm{Span}_{\mathbb{C}}(e_2, \ldots, e_n) \otimes_{\mathbb{C}} \mathcal{K}$ generated by $v_2, \ldots, v_n$. □

Remark 2.8. *We see that* $\mathrm{Gr}_G = \varinjlim \overline{\mathrm{Gr}}_G^\lambda$, $\mathrm{Fl}_G = \varinjlim \overline{\mathrm{Fl}}_G^{\hat{w}}$ *i.e. again see that* Gr_G, Fl_G *are ind-schemes of ind-finite type.*

Now we are ready to return to our Hecke algebras. Note that the I-orbits on Fl_G are in bijection with $\hat{W}$. This is a good sign. Recall that in the finite case in order to construct $\mathcal{H}_q(W)$ we considered the set $\mathrm{SL}_n(\mathbb{F}_q)/B(\mathbb{F}_q) = \mathfrak{B}(\mathbb{F}_q)$.

Question to the audience: which set should we consider in the affine setting?

It is natural then to consider the vector space generated by characteristic functions of $I(\mathbb{F}_q)$-orbits on $\mathrm{Fl}_{\mathrm{SL}_n}(\mathbb{F}_q)$ or more symmetrically the vector space

$\mathbb{C}[I(\mathbb{F}_q)\backslash \mathrm{SL}_n(\mathbb{F}_q((z)))/I(\mathbb{F}_q)]$ generated by characteristic functions of double cosets of $I(\mathbb{F}_q)$ acting on $\mathrm{SL}_n(\mathbb{F}_q)$. We can similarly define the vector space $\mathbb{C}[\mathrm{SL}_n(\mathbb{F}_q[[z]])\backslash \mathrm{SL}_n(\mathbb{F}_q((z)))/\mathrm{SL}_n(\mathbb{F}_q[[z]])]$.

We want to define the convolution product on these vector spaces. Note that $|I(\mathbb{F}_q)| = \infty$ so we can not divide by it in the definition (as we did in the case of $\mathfrak{B}$).

On the other hand recall that for $\mathfrak{B}$ we had

$$\chi_w * \chi_v = \sum_{u\in W} |\mathfrak{B}_w(\mathbb{F}_q) \cap u\mathfrak{B}_{v^{-1}}(\mathbb{F}_q)|\chi_u$$

and we do not divide by anything here. Note that this definition works perfectly well in our setting since $\mathrm{Fl}^{\hat{w}}_{\mathrm{SL}_n}$, $\mathrm{Gr}^{\lambda}_{\mathrm{SL}_n}$ (that are analogs of $\mathfrak{B}_w$) are finite type schemes, hence, the sets of their $\mathbb{F}_q$-points are finite. So we define the convolution product for $\mathrm{Fl}_{\mathrm{SL}_n}$ as follows:

$$\chi_{\hat{w}} * \chi_{\hat{v}} = \sum_{\hat{u}\in \hat{W}} |\operatorname{Fl}^{\hat{w}}(\mathbb{F}_q) \cap \hat{u}\operatorname{Fl}^{\hat{v}^{-1}}(\mathbb{F}_q)|\chi_{\hat{u}}.$$

Note that the sum is finite since if the intersection $\mathrm{Fl}^{\hat{w}}(\mathbb{F}_q)\cap\hat{u}\,\mathrm{Fl}^{\hat{v}^{-1}}(\mathbb{F}_q)$ is nonempty then $I(\mathbb{F}_q)\hat{w}I(\mathbb{F}_q) \cap \hat{u}I(\mathbb{F}_q)\hat{v}^{-1}I(\mathbb{F}_q) \neq \varnothing$ and so $\hat{u} \in I(\mathbb{F}_q)\hat{w}I(\mathbb{F}_q)\hat{v}I(\mathbb{F}_q)$ and there is only finite number of such $\hat{u}$ (actually every such $\hat{u}$ must be $\leqslant$ then $\hat{w}\hat{v}$ with respect to the Bruhat order on $\hat{W}$).

Remark 2.9. *We are implicitly using that* $\mathrm{Fl}_G(\mathbb{F}_q) = G(\mathbb{F}_q((z)))/I(\mathbb{F}_q)$, *recall that* $I(\mathbb{F}_q)$ *is the preimage of* $B(\mathbb{F}_q)$ *under the evaluation at zero homomorphism* $G(\mathbb{F}_q[[z]]) \to G(\mathbb{F}_q)$. *The equality* $\mathrm{Fl}_G(\mathbb{F}_q) = G(\mathbb{F}_q((z)))/I(\mathbb{F}_q)$ *can be shown using Bruhat decompositions of* $G(\mathbb{F}_q((z)))$, $\mathrm{Fl}_G(\mathbb{F}_q)$.

Proposition 2.10. *The algebra*

$$(\mathbb{C}[I(\mathbb{F}_q)\backslash \mathrm{SL}_n(\mathbb{F}_q((z)))/I(\mathbb{F}_q)], *)$$

is generated as a vector space over $\mathbb{C}$ *by elements* $T_{\hat{w}} = \chi_{\mathrm{Fl}^{\hat{w}}_{\mathrm{SL}_n}}$, $\hat{w} \in S_n \ltimes Q$ *subject to the following relations:*

$$T_{\hat{w}_1}T_{\hat{w}_2} = T_{\hat{w}_1\hat{w}_2} \text{ if } l(\hat{w}_1\hat{w}_2) = l(\hat{w}_1) + l(\hat{w}_2),\ \hat{w}_1, \hat{w}_2 \in \hat{W}$$
$$T^2_{s_{\bar{i}}} = q + (q-1)T_{s_{\bar{i}}},\ \bar{i} \in \mathbb{Z}/n\mathbb{Z}.$$

We have

$$\mathbb{C}[\mathrm{SL}_n(\mathbb{F}_q[[z]])\backslash \mathrm{SL}_n(\mathbb{F}_q((z)))/\mathrm{SL}_n(\mathbb{F}_q[[z]])] \simeq (\mathbb{C}Q)^W.$$

Proof. The first part of the theorem is due to Iwahori and Matsumoto (see the original reference [IM] and also the expository paper [HKP]) and can be proved similarly to the finite case (the proof is rather easy when understand the geometry of $\mathrm{Fl}_{\mathrm{SL}_n}$). The second claim is due to Satake (this is so-called Satacke isomorphism). This claim can be considered as a first step towards the formulation of the so-called Langlands conjectures. □

Remark 2.11. *The algebra* $\mathbb{C}Q$ *has two natural bases. One is* $\{\sum_{w\in W} w(\lambda),\ |\ \lambda \in Q^+\}$, *the second one is the basis of characters of irreducible representations* V_λ *of* PGL_n. *Satake isomorphism gives us the third basis in* $\mathbb{C}[Q]^W$ *- the one corresponding to the* *characteristic functions of* $\mathrm{SL}_n(\mathbb{F}_q[[z]])$*-double cosets. All these three bases are distinct. The matrix coefficients relating third basis with the second one have to do with Kazhdan-Lusztig polynomials for* $\hat{W}$*.*

Remark 2.12. *Let us give the formulation of Satake isomorphism for* PGL_n*. Recall that* $\Lambda = \mathbb{Z}^n/\{(k,\dots,k) \,|\, k \in \mathbb{Z}\}$*. We denote by* $\Lambda^+ \subset \Lambda$ *the submonoid consisting of* $[k_1,\dots,k_n]$ *such that* $k_1 \geqslant \dots \geqslant k_n$*. Note that* Λ^+ *is generated by* $n-1$ *elements* $\omega_1,\dots,\omega_{n-1}$*,* $\omega_i = (\underbrace{1,\dots,1}_{i},0,\dots,0)$ *(that is not true for* $Q^+ \subset \Lambda^+$*). Then we have*

$$\mathbb{C}[\mathrm{SL}_n(\mathbb{F}_q[[z]])\backslash \mathrm{SL}_n(\mathbb{F}_q((z)))/\mathrm{SL}_n(\mathbb{F}_q[[z]])] \simeq (\mathbb{C}\Lambda)^W$$

and the algebra above is commutative and freely generated by the characteristic functions $\chi_{\mathrm{Gr}^{\omega_i}_{\mathrm{PGL}_n}(\mathbb{F}_q)}$*.*

Remark 2.13. *Note that* $\mathbb{F}_q((z))$ *is an example of a non-Archimedean local field with* $\mathbb{F}_q[[z]]$ *being its valuation ring. We may replace* $\mathbb{F}_q((z))$ *by any non-Archimedean local field* $\mathcal{K}$ *and* $\mathbb{F}_q[[z]]$ *by its valuation ring* $\mathcal{O} \subset \mathcal{K}$ *and the statement of Proposition 2.10 will still hold.*

Let us now take a look at the Satake isomorphism of Proposition 2.10. Note that the RHS of this isomorphism identifies canonically with the algebra $K_0(\mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n)$, here $\mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n$ is the category of finite dimensional representations of PGL_n over complex numbers and K_0 corresponds to taking K-theory of this category. The algebra structure on $K_0(\mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n)$ is given via tensor product operation. The isomorphism $K_0(\mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n) \xrightarrow{\sim} \mathbb{C}[Q]^W$ sends a class of a representation P to its character $\mathrm{ch}\,P$.

We see that the RHS of the Satake isomorphism can be "categorified" to the category $\mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n$. Note that the LHS also has a categorification that is the category $\mathrm{Per}_{\mathrm{SL}_n(\mathcal{O})}\,\mathrm{Gr}_{\mathrm{SL}_n}$ of perverse $\mathrm{SL}_n(\mathcal{O})$-equivariant sheaves on $\mathrm{Gr}_{\mathrm{SL}_n}$. Category $\mathrm{Per}_{\mathrm{SL}_n(\mathcal{O})}\,\mathrm{Gr}_{\mathrm{SL}_n}$ has a tensor structure $*$ given by the "categorified" version of convolution product. The following proposition is the so-called geometric Satake isomorphism. This is due to Beilinson-Drinfeld-Ginzburg-Lusztig-Mirković-Vylonen.

Proposition 2.14. *There exists an equivalence of tensor categories*

$$(\mathrm{Per}_{\mathrm{SL}_n(\mathcal{O})}\,\mathrm{Gr}_{\mathrm{SL}_n}, *) \simeq \mathrm{Rep}_{\mathrm{f.d}}\,\mathrm{PGL}_n\,.$$

Proof. See [MV]. □

Remark 2.15. *This proposition can be considered as a first step towards the formulation of the so-called geometric Langlands conjectures. Note that the LHS of this isomorphism is some category depending on* SL_n *but the RHS depends on* PGL_n*. The groups* SL_n*,* PGL_n *are so-called Langlands dual groups.*

Remark 2.16. *Some categorification of the first statement of Proposition 2.10 is also known. See* [AB]*.*

3. Affine Grassmannian and flags as moduli spaces of bundles

This section should be covered only if time permits.

Recall that $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$, $\mathrm{Fl}_G = G(\mathcal{K})/I$. Set $D := \operatorname{Spec}\mathcal{O}$, $\mathring{D} := \operatorname{Spec}\mathcal{K}$. One can consider D as a formal neighbourhood of the point $0 \in \mathbb{A}^1$ and $\mathring{D}$ can be considered as a punctured formal neighbourhood of 0.

Proposition 3.1. *Space* Gr_G *is the moduli space of pairs* $(\mathcal{E}_D, \sigma_{\mathring{D}})$ *where* $\mathcal{E}$ *is a principal* G*-bundle on* D *and* $\sigma_{\mathring{D}} \colon \mathcal{E}^{\mathrm{triv}}_{\mathring{D}} \xrightarrow{\sim} \mathcal{E}_{\mathring{D}}$ *is a trivialization of* $\mathcal{E}_D$ *restricted to* $\mathring{D}$.

Space $\mathrm{Fl}_{\mathrm{GL}_n}$ *is the moduli space* $(\mathcal{E}_D, \sigma_{\mathring{D}}, F)$*, where* $(\mathcal{E}_D, \sigma_{\mathring{D}})$ *are the data as above and* F *is a* B*-subtorsor in the fiber* $\mathcal{E}_0$*. For* $G = \mathrm{GL}_n$ *this is the same as the space of triples* $(\mathcal{E}_D, \sigma_{\mathring{D}}, F_\bullet)$ *with* $\mathcal{E}_D$ *being rank* n *vector bundle on* D*,* $\sigma_{\mathring{D}}$ *its trivialization on* $\mathring{D}$ *and* $F_\bullet$ *a flag in the fiber* $\mathcal{E}_0$*.*

Proof. We only prove the first claim. The second claim is an exercise. We also assume that $G = \mathrm{GL}_n$. Recall that $\mathcal{O}$ is a PID so every vector bundle on D is trivial. Let us now fix a trivialization $\sigma_D \colon \mathcal{E}^{\mathrm{triv}}_D \xrightarrow{\sim} \mathcal{E}_D$ of our vector bundle $\mathcal{E}_D$. Then after restricting it to $\mathring{D}$ it together with $\sigma_{\mathring{D}}$ gives us an element of $\mathrm{GL}_n(\mathcal{K})$. Forgetting the choice of σ_D corresponds to considering g modulo $\operatorname{Aut}\mathcal{E}^{\mathrm{triv}}_D = \mathrm{GL}_n(\mathcal{O})$. $\square$

One also has a "global" description of Gr_G Let now C be any smooth curve over $\mathbb{C}$ and pick a point $x \in C$ and set $\mathring{C} := C \setminus \{x\}$.

Proposition 3.2. *Space* Gr_G *is the moduli space of pairs* $(\mathcal{E}_C, \sigma_{\mathring{C}})$ *where* $\mathcal{E}_C$ *is a principal* G*-bundle on* C *and* $\sigma_{\mathring{C}}$ *is a trivialization of* $\mathcal{E}_C$ *restricted to* $\mathring{C}$*.*

Proof. We give the idea of the proof, for the details see for example [G]. Starting from a pair $(\mathcal{E}_C, \sigma_{\mathring{C}})$ we can just restrict it to D to obtain the desired point of Gr_G. In the opposite direction we start from the pair $(\mathcal{E}_C, \sigma_{\mathring{C}})$. Consider now the covering $D \sqcup \mathring{C} \twoheadrightarrow C$. This is a fully faithfull morphism. It then follows from the faithfully flat descent (see for example [Stack, Section 58.16]) that we can glue bundle $\mathcal{E}_D$ with the trivial bundle $\mathcal{E}_{\mathring{C}}$ via the trivialization $\sigma_{\mathring{D}}$ (note that $\mathring{D} = D \times_C \mathring{C}$) and obtain the bundle $\mathcal{E}_C$ on the curve C with the canonical trivialization $\sigma_{\mathring{C}}$. $\square$

Corollary 3.3. *We have* $\mathrm{Gr}_G(\mathbb{C}) = G[z^{\pm 1}]/G[z]$.

Proof. Take $C = \mathbb{P}^1$ and $x = 0 \in \mathbb{P}^1$. Then Gr_G is the moduli space of pairs $(\mathcal{E}_{\mathbb{P}^1}, \sigma_{\mathbb{P}^1\setminus\{0\}})$. We assume that $G = \mathrm{GL}_n$. Then $(\mathcal{E}_{\mathbb{P}^1})|_{\mathbb{A}^1}$ must be trivial (since $\mathbb{C}[\mathbb{A}^1]$ is a PID). We can fix any trivialization $\sigma_{\mathbb{A}^1}$ of $(\mathcal{E}_{\mathbb{P}^1})|_{\mathbb{A}^1}$ and together with $\sigma_{\mathbb{P}^1\setminus\{0\}}$ obtain the gluing function $g \in G[z^{\pm 1}]$ that determines $\mathcal{E}_{\mathbb{P}^1}$ uniquely. Forgetting the choice of $\sigma_{\mathbb{A}^1}$ corresponds to considering g modulo $\operatorname{Aut}(\mathcal{E}^{\mathrm{triv}}_{\mathbb{A}^1}) = G[z]$. $\square$

Another feature that appears in this infinite-dimensional setting is that there are interesting orbits that are opposite to $G(\mathcal{O})$-orbits. Note that in finite case orbits of B and B_- (lower triangular matrices) are isomorphic (conjugate by the longest element of the Weyl group for S_n given by $i \mapsto n+1-i$). In the affine situation the analog of B_- is the group $I_- \subset G[z^{-1}]$ of matrices which value at infinity lies in B_- and we have $G[z^{-1}]$ for $G(\mathcal{O})$. Note that $G[z^{-1}]$ is not isomorphic to $G(\mathcal{O})$. Note for example that $G(\mathcal{O})$ is represented by a scheme that is pro-finite type but $G[z^{-1}]$ is and ind-scheme of ind-finite type. So another interesting thing is the decomposition of Gr_G

into $G[z^{-1}]$-orbits. This decomposition is closely related to the description of Gr_G via bundles.

Proposition 3.4. *We have*

$$\mathrm{Gr}_G = \bigsqcup_{\lambda\in\Lambda^+} G[z^{-1}]\cdot z^\lambda$$

Proof. We assume that $G = \mathrm{GL}_n$. Recall that $\mathrm{Gr}_G = G[z^{\pm1}]/G[z]$. Our goal is to parametrize $G[z^{-1}]\backslash G[z^{\pm1}]/G[z]$. I claim that this set is in bijection with the set of isomorphism classes of rank n vector bundles on $\mathbb{P}^1$. Indeed vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ should be trivial being restricted to $\mathbb{P}^1\setminus\{0\}$, $\mathbb{P}^1\setminus\{\infty\}$ so is determined by its gluing function that is the element of $\mathrm{GL}_n(\mathbb{P}^1\setminus\{0,\infty\}) = \mathrm{GL}_n[z^{\pm1}]$. Moreover two gluing functions define the same vector bundle iff they lie in the same double coset of $G[z^{-1}]\times G[z]$.

It remains to note that by the Grothendieck theorem (one proof is via cohomological) every vector bundle on $\mathbb{P}^1$ is isomorphic to the direct sum $\mathcal{O}(\lambda_1)\oplus\ldots\oplus\mathcal{O}(\lambda_n)$. Moreover assuming that $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n$ we see. $\square$

Remark 3.5. *Let us add here couple remarks about the geometry of* Gr_G*. Ind-scheme* Gr_G *is formally smooth but can not be presented as an inductive limit of smooth schemes of finite type. Ind-scheme* Gr_G *is reduced iff* G *is reductive.*

4. Appendix

We are using the following proposition (proof is an exercise).

Proposition 4.1. *Let $f\colon X\to Y$ be a $\mathbb{C}^\times$-equivariant morphism of smooth affine varieties equipped with an action of $\mathbb{C}^\times$. Assume that the $\mathbb{C}^\times$-actions contract X to the unique fixed point $x\in X$ and contract Y to the $\mathbb{C}^\times$-fixed point $y\in Y$. Assume also that f induces the isomorphism $T_xX\xrightarrow{\sim}T_yY$. Then f is an isomorphism.*

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